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The Malkus theory applied to magnetohydrodynamic turbulent channel flow

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The turbulent flow of a weakly conducting liquid between parallel plates in the presence of a transverse magnetic field is investigated. The form of the mean velocity profile is determined by a series of constraints resulting from the boundary conditions and the Navier–Stokes equations and by the Malkus postulates on the spectrum of the mean vorticity gradient. The width of the transition regions near the walls is derived in terms of the governing dimensionless numbers and this expression is checked, in the asymptotic laminar case, against the well-known Hartmann result. A graphical method, exploiting the relation between the boundary region thickness and the smallest scale of motion defined by the Malkus theory is proposed to determine the *scale* of the velocity profile, i.e. the flow rate in terms of the pressure gradient and the magnetic field strength.

1. Introduction

Few advances have been so far recorded by dimensional analysis in magnetohydrodynamic turbulent channel flow. The existing theories (Hartmann & Lazarus 1937; Murgatroyd 1953; Harris 1960), besides being partly contradictory, are not entirely successful and, indeed, there are probably too many parameters necessary to describe the problem to infer a sensible relation between them without being forced to adopt additional and often dubious assumptions. Hartmann & Lazarus attempted to separate their results into a 'turbulence-damping effect' typical of turbulent flows and a 'viscosity effect' similar to that found in laminar flows. The procedure by which they effected this separation involved several delicate assumptions and, in particular, implied that the 'turbulence-damping effect' is independent of the conductivity. Murgatroyd (1953) proposed instead a dimensional theory based on the hypothesis that the friction factor does not depend on the viscosity ν. This hypothesis was severely criticized by Harris who pointed out that the friction factor, in laminar hydrodynamic and hydromagnetic flows (between insulating walls) varies as ν and $\nu^{\frac{1}{2}}$ respectively and is a function of $\ln \nu$ in turbulent hydrodynamic flow. Assuming that, at points near the boundaries, the mean velocity profile is independent of the channel span and that, at points near the centre, the local structure of the mean flow is independent of the viscosity, Harris obtained the velocity profile in terms of two unknown functions. One of these, he claimed—independent of the viscosity—gave the distortion in the velocity profile caused by hydromagnetic effects, the other—independent of the channel span—represented over the central part of the channel a constant addition to the mean velocity. For reasons difficult to understand, the latter was ultimately neglected in the expression of the friction factor and Harris added the requirement that this expression reduce to the known result of ordinary turbulence when the magnetic field strength is zero. The comparison with the experiments of Murgatroyd showed a satisfactory agreement only for moderate field strengths (the theory explains well the departure from purely hydrodynamic turbulence as a result of the application of a weak field) and, indeed, it is likely that in partly inhibited turbulence, at higher values of the applied magnetic field, the viscosity plays an important role.

The limited success of the dimensional arguments suggest the resort to a totally different approach like the Malkus theory. Since its publication in 1954 and 1956, the Malkus theory of (ordinary) non-homogeneous turbulence has been much debated. It is based on a series of assumptions which are by no means indisputable, but which lead to expressions of the mean temperature distribution in convective turbulence and of the mean velocity profile in shear turbulence in excellent agreement with the experimental observations. Later, Malkus reformulated his theory (Malkus 1961*a*, *b*) to emphasize the distinction between the fundamental assertions and the consequences of the mathematical formulation of the problem, and the complexity of the numerical analysis in which the basic assumptions were initially immersed.

The following are the assumptions on which the theory is based (Malkus 1956) in the particular context of turbulent channel flow:

1. There is no point of inflexion in the mean velocity profile.

2. There is a smallest scale of motion which contributes to the transport of momentum.

3. The smallest scale of motion is the smallest scale to which the mean profile is unstable on laminar theory.

4. The total dissipation rate is greater for the actual flow than for any other flow with the same flux and satisfying the condition (1) and the boundary conditions on the walls.

As pointed out by Townsend (1961), these assumptions fall into two categories. The first consists of the 'kinematic assumptions' 1 and 2. In addition to the more recent papers by Malkus, a very persuasive discussion of the arguments in their favour has been presented by Spiegel at Marseille (1961) in the context of convective turbulence. They have been further discussed in the context of the problem of stability of MHD channel flow in a transverse field by Nihoul (1966).

The 'dynamical' hypotheses of the second category are, on the other hand, much more difficult to understand. In his original paper on shear turbulence, Malkus suggested that the maximization of the dissipation rate would leave less energy for disturbances of the mean and increase the stability of the mean velocity profile, and he invoked a series of thermodynamical arguments in support. He later refined this approach and proposed that the stable solutions of turbulent shear flow were those of minimum dissipation for fixed momentum flux and those of maximum dissipation for fixed mean flow (Malkus 1961; Veronis 1961). In shear flow in a channel, where the temperature is nearly the same everywhere, the mechanical dissipation is roughly equivalent to the entropy production. Malkus's principle has then some contact with the ideas of Glansdorff & Prigogine (1954, 1963, 1964)† and its dual aspect may imply that the stable statistically steady turbulent state is a saddle-point in the surface of entropy production; this saddle-point being a minimum with respect to variations subject to fixed boundary fluxes and a maximum with respect to fluctuations resulting in variations of the boundary fluxes. The prospects of this meeting between the Malkus theory and familiar ideas of statistical mechanics are promising but, at the present stage, one must admit that a certain dubiety remains regarding the physical concept at the origin of Malkus's variational principle. However, Townsend (1961) has pointed out that the form, though not the scale, of the velocity distribution was probably a consequence of the first two assumptions and he succeeded in obtaining the appropriate velocity profile by considering the best way of approximating to the asymptotic distribution with a finite series giving non-positive values for the mean vorticity gradient.

In the following, the Malkus theory is applied to MHD turbulent channel flow and it is shown that assumptions 1 and 2 are sufficient, in this case also, to determine the *shape* of the mean velocity profile. Assuming then that the magnetic field is sufficiently large to determine the width of the transition regions, through the modifications it produces in the boundary constraints, we derive the expression of the smallest scale in terms of the dimensionless numbers of the problem and, with arguments related to the third hypothesis, we deduce a relation between these dimensionless numbers. This corresponds to estimating say, the Reynolds number in terms of the others, i.e. determining the *scale* of the profile. These results are found to agree with those of Hartmann in the asymptotic laminar case and with the experimental measurements of Murgatroyd.‡

2. The fundamental equation of MHD turbulent parallel flow

We examine the flow of an incompressible conducting fluid between two parallel plates (at $x_2 = \pm L$). The x_1-axis is taken in the direction of the flow. A uniform magnetic field $\mathbf{b}_0$ is applied perpendicular to the plates. We assume that the magnetic Reynolds number is small and neglect the induced magnetic field $\mathbf{b}$ as compared with $\mathbf{b}_0$. In terms of the velocity $\mathbf{v}$ and the 'local' Alphen velocity $\mathbf{h} = (\mu\rho)^{-\frac{1}{2}}\mathbf{b}$, the basic equations may be written (see, for instance, Nihoul 1963)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} + \frac{1}{\rho}\nabla p^* = \mathbf{h}_0.\nabla\mathbf{h} + \nu\nabla^2\mathbf{v}, \tag{1}$$

$$\mathbf{h}_0.\nabla\mathbf{v} + \lambda\nabla^2\mathbf{h} = 0, \tag{2}$$

$$\nabla.\mathbf{v} = \nabla.\mathbf{h} = 0, \qquad (3), (4)$$

† Malkus (1961) pointed out certain differences between his principle and Prigogine's theorem in its original form but they are probably not irreconcilable, especially in view of the most recent work of Prigogine (1964).

‡ Since no use is made of assumption 4, there are a few numerical factors of order unity which cannot be better specified (although the asymptotic agreement with the laminar case gives much information). However, in view of the very little experimental data available (only the flow rate has been measured) it does not seem desirable to undertake the enormous numerical work which the application of the variational principle would require.

where $p^* = p+\rho\mathbf{h}_0.\mathbf{h}$ is the 'total' pressure, ρ is the density, ν the kinematic viscosity, $\lambda = (\mu\sigma)^{-1}$ the magnetic viscosity, μ the permeability and σ the conductivity. The term $\partial\mathbf{h}/\partial t$ which might be expected in (2) is negligible provided the ohmic diffusion time scale is small compared with the fluctuating velocity time scale.

Denoting ensemble averages by $\langle\ \rangle$, let

$$\langle\mathbf{v}\rangle = \mathbf{u}(x_2) = (u,0,0), \quad \mathbf{v} = \mathbf{u}+\mathbf{w}, \tag{5}$$

$$\langle\mathbf{h}\rangle = \mathbf{a}(x_2) = (a,0,0), \quad \mathbf{h} = \mathbf{a}+\mathbf{c}, \tag{6}$$

$$\langle p\rangle = m^*, \quad p^* = m^*+r^*. \tag{7}$$

Substituting in (1) and (2) and averaging, we obtain

$$\nu\frac{d^2u}{dx_2^2}+h_0\frac{da}{dx_2}-\frac{1}{\rho}\frac{\partial m^*}{\partial x_1} = \frac{d}{dx_2}\langle w_1w_2\rangle, \tag{8}$$

$$-\frac{1}{\rho}\frac{\partial m^*}{\partial x_2} = \frac{d}{dx_2}\langle w_2^2\rangle, \tag{9}$$

$$h_0\frac{du}{dx_2}+\lambda\frac{d^2a}{dx_2^2} = 0. \tag{10}$$

Equations (9) and (10) may be integrated once, giving

$$h_0u+\lambda\frac{da}{dx_2} = -(\mu\rho)^{-\frac{1}{2}}\epsilon_3, \tag{11}$$

$$\langle w_2^2\rangle+\frac{m^*}{\rho} = \frac{m_0^*}{\rho} = \frac{m_0}{\rho}, \tag{12}$$

where ϵ_3 is the constant electric field in the x_3-direction and where the subscript 0 denotes the value at the wall.

Defining for the convenience of later discussions (a bar refers to the average over x_2)

$$\tau_0^{\frac{1}{2}} = [-(L/\rho)\,\partial m_0/dx_1]^{\frac{1}{2}} \quad \text{(friction velocity)}, \tag{13}$$

$$\eta = \epsilon_3/\bar{u}b_0, \tag{14}$$

and eliminating a and m between (8), (9) and (10), we obtain the fundamental equation for u,

$$\nu\frac{d^2u}{dx_2^2}-\frac{h_0^2}{\lambda}u = \eta\frac{h_0^2}{\lambda}\bar{u}-\frac{\tau_0}{L}+\frac{d}{dx_2}\langle w_1w_2\rangle. \tag{15}$$

3. Mathematical formulation of the Malkus theory

The mean vorticity gradient being everywhere of the same sign, we may write

$$d^2u/dx_2^2 = -Ag^2, \tag{16}$$

where A is a constant and g a real function. Following Malkus, we expand g in series of orthogonal functions ϕ_n and truncate the series at the n_0th term where $n_0 \gg 1$:

$$g = \sum_0^{n_0} Y_n\psi_n. \tag{17}$$

Each of the ψ_n's is characterized by a wave-number κ_n such that $\psi_n' \sim \kappa_n \psi_n$.† We now make the additional assumption that, away from the boundary (farther than one local wave-length), the leading term in (17) is of order κ_0 larger than all other terms, providing an asymptotic 'law' of order κ_0 larger than the rest (see equation (41)). This requirement is equivalent to the condition that the coefficient Y_n be smooth or equivalently that it be properly represented by a finite polynomial in κ_n/κ_0 (Malkus 1961*b*).

Symmetry and boundary constraints

The velocity profile must be symmetrical with respect to the plane $x_2 = 0$. (In the following we shall consider only the region $0 \leqslant x_2 \leqslant L$.) Since

$$w_1 = w_2 = w_3 = 0 \quad \text{and} \quad \partial w_2/\partial x_2 = 0$$

(from $\nabla . \mathbf{v} = 0$) at the boundary surfaces,

$$\langle w_1 w_2 \rangle = \frac{d\langle w_1 w_2 \rangle}{dx_2} = \frac{d^2\langle w_1 w_2 \rangle}{dx_2^2} = 0 \quad \text{at} \quad x_2 = \pm L. \tag{18}$$

We assume that the application of an electric and a magnetic field is not sufficient to reverse the flow. Hence, the velocity being zero at the boundary, the constant A in (16) must be positive. From this requirement, and from equations (18) and (15), it follows that

$$Z - \eta M^2 \geqslant 0, \tag{19}$$

where $Z = \tau_0 L/\bar{u}\nu$, $M^2 = b_0^2 L^2/\nu\lambda$ (M is the Hartmann number), and we may remove the arbitrariness in A by defining

$$A = (\bar{u}/L^2)\,(Z - \eta M^2). \tag{20}$$

Combining (15) and (18) and introducing the non-dimensional co-ordinate $y = x_2/L$, we obtain the following boundary and symmetry constraints

$$(g^2)_1 = 1, \tag{21}$$

$$\int_0^1 g^2\,dy = \frac{Z - (1+\eta)M^2}{Z - \eta M^2}, \tag{22}$$

$$\int_{-1}^1 yg^2\,dy = 0, \tag{23}$$

and

$$\left(\frac{dg^2}{dy}\right)_1 = M^2 \int_0^1 g^2\,dy. \tag{24}$$

† κ_n is not necessarily equal to n. It is equal to n in the case of circular functions but, in the case of Legendre polynomials, for instance, near the boundary, we have

$$\psi_n' \sim n(n+1)\psi_n'$$

and the characteristic wave number is the eigenvalue $n(n+1) \sim n^2$ for large n. The choice of one type of orthogonal function or another is determined by the boundary conditions and we shall see that circular functions are quite appropriate in the absence of a transverse magnetic field but that, in magnetoturbulence, circular functions would not permit the satisfaction of one of the boundary constraints and that Legendre polynomials are required.

For consistency of equations (19) and (22),

$$Z \geqslant (1+\eta) M^2. \tag{25}$$

Condition (25) imposes a lower bound on the (non-dimensional) shear stress as a function of the Hartmann number. The same inequality holds for laminar as well as turbulent flows. (The boundary constraints are the same.) It is readily seen (Hartmann 1937) that the laminar flow satisfies (25) with equality in the limit $M \to \infty$.

Determination of the function g

In the absence of a transverse magnetic field, (24) is replaced by $(dg^2/dy)_1 = 0$. Hence a convenient orthonormal set ψ_n which automatically satisfies (23) and (24) is

$$\psi_n(\Phi) = 2^{\frac{1}{2}} \cos 2n\Phi \quad (\psi_0 = 1),$$

where $\Phi = \frac{1}{2}\pi(1-y)$. This set is inappropriate in the magnetohydrodynamic case, since the terms of (17) cannot then individually satisfy the condition (24), which actually represents the main effect of the transverse magnetic field on the flow. As shown in the following, if the Hartmann number is large enough, the magnetic field determines the structure of the boundary layers through this condition. It is readily found that the Legendre polynomials of even order are the appropriate orthogonal functions which satisfy the condition of symmetry (23). This choice is guided by the boundary constraints. Hence if some result, later, appears as a property of the Legendre polynomials, we may be confident that it represents a genuine property of the flow. We therefore adopt the expansion

$$g = \sum_{0}^{\frac{1}{2}n_0} Y_n P_{2n}(y). \tag{26}$$

To satisfy the requirement of smoothness of the coefficient Y_n, we write it in the form of a finite polynomial. Let

$$Y_n = \frac{4n+1}{n_0} \sum_{s=0}^{s=S} a_s \left[\frac{2n(2n+1)}{n_0^2}\right]^s, \tag{27}$$

where S is any large integer less than n_0 (Malkus 1961*b*). n_0 is assumed very large and factors like n_0+1 are replaced by n_0. The factor $(4n+1)$ is found necessary in order to sum the series (26) (Malkus 1961*b*). Substituting (27) into (26) and performing these summations, we obtain g in terms of ultraspherical Jacobi polynomials $C^{\lambda}_{n_0}$ (see the appendix). Since n_0 is large, we may restrict attention to the first terms in their asymptotic expansions. Now it is a well-known property of asymptotic expansions that they may be different in different domains. In the present case, distinct asymptotic behaviour is found near the boundary where $1-y^2 \sim O(1/n_0^2)$ and away from the boundary, in the core where $1-y^2 \gg 1/n_0^2$.

The function g in the core

In the core, setting $y = \cos\theta$, we have (see the appendix)

$$g_c \simeq \frac{\alpha}{n_0} C^{\frac{3}{2}}_{n_0} \simeq \frac{\alpha}{n_0} \frac{(\frac{1}{2}n_0)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})(\sin\theta)^{\frac{3}{2}}} \cos\left[(n_0+\tfrac{3}{2})\theta - \tfrac{3}{4}\pi\right], \tag{28}$$

where

$$\alpha = \sum_{0}^{S} a_s. \tag{29}$$

The function g in the transition region

In the transition region, setting $y = 1-2\zeta/n_0^2$, we obtain

$$g_b \sim n_0[\beta J_0(2\zeta^{\frac{1}{2}})+\gamma J_2(2\zeta^{\frac{1}{2}})], \tag{30}$$

where J denotes the Bessel function and where

$$\beta = \frac{1}{2}\sum_0^S \frac{a_s}{s+1}, \quad \gamma = \sum_0^S \frac{a_s}{(s+1)(s+2)}. \tag{31), (32}$$

The first boundary constraint

Equation (21) implies that

$$\left[\sum_0^{\frac{1}{2}n_0} Y_n\right]^2 = 1. \tag{33}$$

Substituting the expression of Y_n given by equation (27) and writing

$$2\sum_0^{\frac{1}{2}n_0}(2n)^k \sim \frac{n_0^{k+1}}{k+1} \quad \text{for} \quad n_0 \gg 1, \tag{34}$$

we obtain the condition

$$\beta = 0. \tag{35}$$

Indeed, g being $O(1)$ at the boundary, the term proportional to n_0 must vanish at the boundary. Going back to equation (30), we see that this implies $\beta = 0$.

J_2 and all higher-order Bessel functions being zero at the wall, the next term in the asymptotic expansion of g (the term which is $O(1)$ in the series in powers of n_0 and which has been omitted in (30)) becomes the leading term at the wall. However, $J_2(2\zeta^{\frac{1}{2}})$ behaving like ζ near the origin, this term is important only over a range of ζ of order $1/n_0$, i.e. within a distance from the boundary of order $1/n_0^3$. Hence the effect of this term on the mean velocity profile in the transition region—and, *a fortiori*, in the core—is entirely negligible. Now the third boundary condition can be written

$$\left[4g^2\left(\frac{dg}{dy}\right)^2\right]_1 = M^4\left[\frac{Z-(1+\eta)M^2}{Z-\eta M^2}\right]^2,$$

or, since $(g^2)_1 = 1$,

$$4\left(\frac{dg}{dy}\right)^2 = M^4\left[\frac{Z-(1+\eta)M^2}{Z-\eta M^2}\right]^2. \tag{36}$$

In the derivative of g, the dominant contribution comes from the Bessel functions (there is a contribution $\sim n_0 J_0$ at the wall). Hence, if we use (36) instead of (24), we may completely disregard the next term in the expression of g_b and write, in the transition region

$$g_b \sim n_0\gamma J_2(2\zeta^{\frac{1}{2}}). \tag{37}$$

The second boundary constraint

Combining (22), (28) and (37) gives

$$\frac{\alpha^2}{n_0^2}\int_0^{1-\Sigma} C_{n_0}^{\frac{3}{2}}(y)\,C_{n_0}^{\frac{3}{2}}(y)\,dy + n_0^2\gamma^2\int_{\substack{\text{over the}\\\text{trans. reg.}}} J_2^2 dy = \frac{Z-(1+\eta)M^2}{Z-\eta M^2},$$

where $n_0^{-2} \ll \Sigma \ll 1$. In the range $0 < y < 1-\Sigma$, $C_{n_0}^{\frac{3}{2}}$ behaves like $n_0^{\frac{1}{2}}$ and the first integral may be neglected as compared with the second, which is readily transformed by a change of variable into

$$\int_{\substack{\text{over the}\\ \text{trans. reg.}}} J_{\frac{3}{2}}^2\, dy = \frac{2}{n_0^2}\int_0^{\xi} J_{\frac{3}{2}}^2(2\zeta^{\frac{1}{2}})\, d\zeta,$$

where $\xi = O(1)$. Hence

$$\frac{Z-(1+\eta)\,M^2}{Z-\eta M^2} = \tfrac{1}{4}\gamma^2\Gamma^2, \tag{38}$$

$$\Gamma^2 = 8\int_0^{\xi} J_{\frac{3}{2}}^2(2\zeta^{\frac{1}{2}})\, d\zeta = 4\xi[J_{\frac{3}{2}}^2(2\xi^{\frac{1}{2}}) - J_1(2\xi^{\frac{1}{2}})\, J_3(2\xi^{\frac{1}{2}})] \tag{39}$$

and $\Gamma^2 = O(1)$ for $\xi = O(1)$.

The third boundary constraint

Near the wall

$$dg/dy = dg/d\sigma(-n_0^2/\sigma),$$

where

$$y = 1-\frac{2\zeta}{n_0^2} = 1-\frac{\sigma^2}{2n_0^2}, \quad (\sigma = 2\zeta^{\frac{1}{2}}).$$

Hence (36) yields

$$4n_0^6\gamma^2\left[\frac{J'(\sigma)}{\sigma}\right]^2 = M^4\left[\frac{Z-(1+\eta)\,M^2}{Z-\eta M^2}\right]^2. \tag{40}$$

In deriving the formulae (37), (38) and (39), the implicit assumption was made that γ is not zero. We now see that this corresponds to requiring that the Hartmann number be 'sufficiently large' (to account for the order of magnitude of the left-hand side of (40)). Now, the experiments of Murgatroyd (figure 1) suggest that between the laminar domain and the highly turbulent domain occurring for small values of M/R, there is a region—which we may agree to call the 'inhibition region' (Nihoul 1966)—where the little sensitivity of the friction factor to the Reynolds number indicates very likely the prevailing influence of the magnetic field and we may thus expect that, by the third boundary constraint, the magnetic field is the cogent factor in the determination of κ_0 and the boundary layer thickness. This suggests that we may perhaps take the lower limit of the inhibition region as an estimate of the minimum value of M/R for the derivation leading to equation (40) to be valid. Restricting the analysis to the inhibition region, we may substitute (38) into (40). We obtain

$$n_0^6 = \Gamma^2 M^4\frac{Z-(1+\eta)\,M^2}{Z-\eta M^2} = \kappa_0^3. \tag{41}$$

As expected, this formula determines κ_0 in terms of the dimensionless numbers of the problem within a factor of order 1.

Let us go back to equation (17) and the fundamental assumption that g may be adequately represented by a truncated polynomial expansion. In the case of a flat velocity profile with narrow regions of transition near the boundaries—and the fact that we do get such a profile constitutes a test of consistency of one essential postulate of Malkus's theory—the vorticity gradient is significant only over the boundary regions. Since the width of these regions is $O(L/\kappa_0)$, all orthogonal functions corresponding to wave-numbers larger than κ_0 oscillate many times over the regions where g is important and this implies that, in an infinite series, their coefficients (which are given by an integral of $g\psi_n$) would be very small. Hence, we may be confident that the truncated expansion (17) is an adequate description and that the effect of a 'tail' beyond κ_0, if taken into account (by a perturbation method, for instance) would modify very little the conclusions above. (This is discussed in more detail in Nihoul 1966.)

The condition of 'normalization'

Let $v = u/\bar{u}$ and $y = x_2/L$. Equation (16) may be written, with the help of (20)

$$d^2v/dy^2 = -[Z-\eta M^2]g^2. \tag{42}$$

Hence, since $dv/dy = 0$ at $y = 0$ and $v = 0$ at $y = \pm 1$,

$$\frac{dv}{dy} = -[Z-\eta M^2]\int_0^y g^2\,dy, \tag{43}$$

and

$$v = -[Z-\eta M^2]\int_1^y dy\int_0^y g^2\,dy. \tag{44}$$

Integrating by parts and averaging over y, we get

$$4 = [Z-\eta M^2]\int_{-1}^{1}(1-y^2)\,g^2\,dy. \tag{45}$$

We divide the integral into an integral over the core and an integral over the transition regions. If $1/\kappa_0 \leqslant \Sigma \leqslant 1$, we have, from (28) and (37)

$$4 = [Z-\eta M^2]\left\{\frac{\alpha^2}{n_0^2}\int_{-(1-\Sigma)}^{1-\Sigma}(1-y^2)\,C^{\frac{3}{2}}_{n_0}C^{\frac{3}{2}}_{n_0}\,dy + 16\gamma^2/n_0^2\int_0^\xi J_2^2(2\zeta^{\frac{1}{2}})\,\zeta\,d\zeta\right\}. \tag{46}$$

In the first integral, we may extend the range of integration from -1 to 1. We make an error less than $1/n_0$. Indeed the maximum values of the Jacobi polynomials occur at the wall and it may be shown (Tricomi 1955) that

$$|C^{\frac{3}{2}}_{n_0}(y)| \leqslant |C^{\frac{3}{2}}_{n_0}(1)| = \tfrac{1}{2}(n_0+2)(n_0+1) \sim n_0^2.$$

In the transition regions, we have

$$|(1-y^2)\,dy| \leqslant O(n_0^{-4}).$$

Hence, since $C^{\frac{3}{2}}_{n_0}$ behaves like $n_0^{\frac{1}{2}}$ in the core,

$$\int_{\substack{\text{over the}\\ \text{trans. reg.}}}(1-y^2)\,C^{\frac{3}{2}}_{n_0}C^{\frac{3}{2}}_{n_0}\,dy\bigg/\int_{\substack{\text{over the}\\ \text{core}}}(1-y^2)\,C^{\frac{3}{2}}_{n_0}C^{\frac{3}{2}}_{n_0}\,dy \leqslant n_0^{-1}.$$

Now the Jacobi polynomials satisfy (Tricomi 1955)

$$\int_{-1}^{1}(1-y^2)^{\lambda-\frac{1}{2}}\,C^{\lambda}_m\,C^{\lambda}_m\,dy = \frac{\pi 2^{1-2\lambda}\Gamma(m+2\lambda)}{m!\,(m+\lambda)\,[\Gamma(\lambda)]^2}. \tag{47}$$

Substituting in (46), we obtain

$$4 = [Z-\eta M^2]\left[\frac{\alpha^2}{n_0}+\frac{\gamma^2\Gamma^2}{n_0^2}\Delta^2\right], \tag{48}$$

where
$$\Delta^2 = \frac{16}{\Gamma^2}\int_0^\xi J_{\frac{2}{3}}^2(2\zeta^{\frac{1}{2}})\,\zeta\,d\zeta = 2\int_0^\xi d\zeta\left[1-\frac{8}{\Gamma^2}\int_0^\xi J_{\frac{2}{3}}^2(2\zeta^{\frac{1}{2}})\,d\zeta\right]. \tag{49}$$

Combining (48) and (38), we have finally

$$\frac{\alpha^2}{n_0}(Z-\eta M^2) = 4[1-(\Delta^2/n_0^2)(Z-(1+\eta)M^2)],$$

i.e.
$$(\alpha^2/n_0)(Z-\eta M^2) = 4(1-\upsilon_\delta), \tag{50}$$

where υ_δ denotes the value of the dimensionless velocity $\upsilon = u/\bar{u}$ at the limit of the transition region
$$\upsilon_\delta = (\Delta^2/n_0^2)[Z-(1+\eta)M^2],$$
as shown below.

Velocity profile in the transition region

From (44),
$$\upsilon = [Z-\eta M^2]\int_y^1 dy\left\{\int_0^1 g^2\,dy - \int_y^1 g^2\,dy\right\}, \tag{51}$$

i.e.
$$\upsilon = \frac{Z-(1+\eta)M^2}{\kappa_0}\int_0^\zeta 2\,d\zeta\left[1-\frac{8}{\Gamma^2}\int_0^\zeta J_{\frac{2}{3}}^2(2\zeta^{\frac{1}{2}})\,d\zeta\right]$$
$$= \frac{\upsilon_\delta}{\Delta^2}\int_0^\zeta 2\,d\zeta\left[1-\frac{8}{\Gamma^2}\int_0^\zeta J_{\frac{2}{3}}^2(2\zeta^{\frac{1}{2}})\,d\zeta\right]. \tag{52}$$

With
$$\upsilon_\delta = \Delta^2\frac{Z-(1+\eta)M^2}{\kappa_0} = \frac{\Delta^2}{\Gamma^{\frac{2}{3}}M^{\frac{4}{3}}}[Z-(1+\eta)M^2]^{\frac{2}{3}}[Z-\eta M^2]^{\frac{1}{3}}. \tag{53}$$

The velocity increases monotonically from $\zeta = 0$ to its maximum value υ_δ at the upper limit of the transition region.

Velocity defect law in the core

Substituting (50) in (28), we have, from (44),

$$\upsilon_{\max} - \upsilon = 4(1-\upsilon_\delta)\int_0^y dy\int_0^y (C_{n_0}^{\frac{3}{2}}/n_0^{\frac{1}{2}})^2\,dy. \tag{54}$$

At large values of n_0, the Jacobi polynomials may be replaced by their asymptotic expansions. Hence

$$\upsilon_{\max} - \upsilon \sim \frac{4}{\pi}(1-\upsilon_\delta)\int_\theta^{\frac{1}{2}\pi}\sin\theta\,d\theta\int_\theta^{\frac{1}{2}\pi}\frac{1+\sin(2n_0+3)\theta}{\sin^2\theta}\,d\theta$$
$$\sim \frac{4}{\pi}(1-\upsilon_\delta)(1-\sin\theta). \tag{55}$$

This expression is consistent with the fact that, if the profile is very flat, on one hand, the difference between υ and $\upsilon_{\max}$ must be very small over the whole core, while on the other hand, the velocity at the upper limit of the transition layer is very nearly equal to the mean velocity, i.e. $\upsilon_\delta \sim 1$.

Width of the transition layer

Within a factor of order unity,

$$\delta \sim \frac{1}{\kappa_0} \sim M^{-\frac{4}{3}}\left[\frac{Z-(1+\eta)M^2}{Z-\eta M^2}\right]^{-\frac{1}{3}}. \tag{56}$$

Here we reach a point where we can test our theory. For when the Hartmann number is sufficiently large, the flow becomes laminar. The width of the transition region, in the limiting case, must still be given by equation (41). Substituting (Hartmann 1937) $Z = (1+\eta)M^2+M$, we get

$$\delta_L \sim M^{-1}, \tag{57}$$

in agreement with the well-known Hartmann result. This is some evidence in favour of the present analysis.

4. The scale of the mean velocity profile

Let a slight perturbation occur in some part of the flow; it is a wave packet obtained by superposing a series of components of the form

$$e^{i\alpha(x-ct)}\,\Phi(y). \tag{58}$$

The largest value of the derivative of Φ will occur in the sharp shear zone at the boundary and this will determine the smallest scale of motion (Malkus 1956). Restricting attention to the first term in the asymptotic expansion of Φ in the complex y-plane, we have (Wasow 1948)

$$\kappa_0 \sim (2\alpha_c Rc)^{\frac{1}{2}}, \tag{59}$$

where R is the Reynolds number. Now α_c, the down-stream wave number, bears some (probably fixed) relation to κ_0, the cross-stream wave number, say

$$\kappa_0 = r\alpha_c. \tag{60}$$

From (52),

$$c = \kappa_0^{-1}[Z-(1+\eta)M^2]\,\hat{c}, \tag{61}$$

where

$$\hat{c} = \int_0^{\xi_c} 2\,d\zeta\left\{1-\frac{8}{\Gamma^2}\int_0^{\zeta} J_2^2(2q^{\frac{1}{2}})\,dq\right\}.$$

Hence,

$$\kappa_0^2 \sim 2r^{-1}\hat{c}[Z-(1+\eta)M^2]R. \tag{62}$$

Following Malkus, we now define a 'boundary Reynolds number'

$$R_B = su_s/\nu, \tag{63}$$

where s is a distance proportional to κ_0^{-1}, say $s = \kappa_0^{-1}$, and u_s is the velocity that would be due to the initial gradient at that distance from the boundary. So defined, equation (63) may be written

$$R_B = R[Z-(1+\eta)M^2]/\kappa_0^2 \tag{64}$$

from (22) and (43).

Combining (41) and (62), we obtain the required relation between the dimensionless numbers R, Z and M. This relation, however, still contains unspecified parameters r, $\hat{c}$, ξ which must be determined by solving the eigenvalue problem of the Orr–Sommerfeld equation, expressing the condition of marginal stability ($\operatorname{Im} c = 0$) and the requirement of maximum dissipation rate (or minimum Reynolds number for constant dissipation rate, Malkus 1956). To avoid this mathematically complex variational problem we shall give here a different approach making use of the stability requirements on the smallest scale in a more intuitive form and of dimensional arguments predicting the variation of the boundary Reynolds number.

For simplicity, we shall, from now on, restrict attention to the case $\eta = -1$ (which is the situation considered by Hartmann, Murgatroyd, Lock and Harris). In equation (64), κ_0 and one of the dimensionless numbers (Z, say) are (so far unknown) functions of the other two dimensionless numbers. The boundary Reynolds number will thus turn out finally as a function of M and R and we may speculate that it will actually be a function of the sole combination M/R as it probably does not depend on the channel span. So let

$$R_B = (R/M)\, q(M/R). \tag{65}$$

We expect that the function q will be $O(M/R)$ for small M and will approach unity fairly rapidly as the Hartmann number increases. From (41), (64) and (65), we have, in the case $\eta = -1$,

$$Z^{\frac{1}{3}}[M^2+Z]^{\frac{2}{3}} = q(M/R)\, M^{\frac{5}{3}}. \tag{66}$$

At sufficiently large Hartmann number, we expect $q \approx 1$, and equation (66) provides a first estimate of the flow rate in terms of the shear stress and magnetic field strength. We observe that, if $M^2/Z \gg 1$, as is the case in the inhibition region, (66) gives Z/R as a function of M/R only, *in accordance with the main conclusion drawn by Murgatroyd from his experiments*.† We shall now try to improve this estimate and we shall, for this, rely on the results obtained by Lock. Our approach is, in a way, the graphic equivalent of the numerical method of Malkus.

One of the main conclusions reached by Lock (1956) is that, given a neutral stability curve $\alpha(R)$ corresponding to a flat-topped profile with narrow boundary layers, the effect of reducing the width of these layers by superimposing an appropriate transverse field (on the one which pre-existed) is merely a translation of the $\alpha(R)$ curve or, in other words, a change of units along the α and R axes, the new units being proportional to M, $s_1 M$ and $s_2 M$, say, respectively. If the reduction of the boundary layer thickness is produced by superimposed turbulence, we may speculate that the neutral stability curve will undergo the same sort of translation but its amount and direction will presumably be set by the Reynolds number and the new units along the axes will probably be proportional to R, $s_1' R$ and $s_2' R$, say (Nihoul 1966). This will happen to the neutral stability curve corresponding to any value of M and thus—as a result of the appearance of turbulence producing a further flattening of the velocity profile—we foresee that the curve $\alpha_c(M)$ will itself be translated, the amount and the direction of the translation being set by the value of the Reynolds number. This suggests that the curve $\alpha_c(M)$ drawn by Lock is actually universal if appropriate units $s_1'' R$, $s_2'' R$ are used along the axes. Since $r\alpha_c = \kappa_0$, this curve may also be regarded as a representation of κ_0/R as a function of M/R. (We assume that r is constant, at least over the inhibition region and we absorb it in the scaling factor.) On this diagram, the linear part will presumably correspond to laminar flows (we know

† In his paper, Murgatroyd suggested that this observation could support his dimensional law according to which Z/R was a function of $M/Rn^{\frac{1}{2}}$ only, as all his experiments were carried out at constant value of n ($n = \nu/\lambda$).

that in the laminar case $\kappa_0 \sim M$). If we impose the condition that the transition between turbulent and laminar regimes occurs at the value of M/R observed by Murgatroyd, we have sufficient information to determine the scaling factors on the axes. It is not necessary to do it explicitly. If we graduate the x-axis in M/R such that the transition to the laminar law occurs at the value reported by Murgatroyd, the ratio of the ordinate of the curve α_c and the ordinate of its 'asymptote' gives κ_0/M for all values of M/R. Hence, we have a graphic method of determining the smallest scale of motion and this dispenses with the necessity of determining it by heavy numerical calculations.

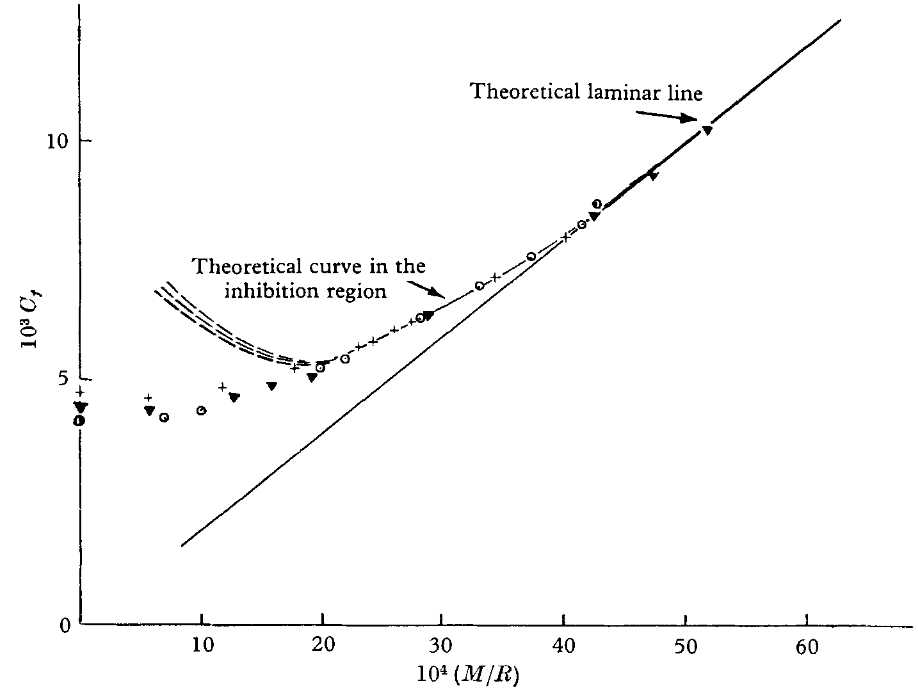

FIGURE 1. Friction factor ($C_f = 2Z/R$) versus M/R. Experimental points (Murgatroyd 1953) ○, $R = 3\times10^4$; ▼, $R = 2{\cdot}5\times10^4$; ×, $R = 2\times10^4$.

For each value of κ_0/M measured on the diagram, we may, by equation (41) calculate R for given F and M (i.e. the flow rate as a function of the shear stress and the magnetic field strength). To allow the comparison with the experiments of Murgatroyd, we have calculated the friction factor ($C_f = 2Z/R$) for different values of M/R and R and we have found an excellent agreement between the calculated values and the measurements (figure 1).

Clearly, the graphic method described here allows for a certain margin of quantitive error in the scaling identification and especially in the location of the transition point between turbulent and laminar flow on Lock's diagram (which is made particularly difficult by the smoothness of contact between the two curves). If this point of transition is adroitly located, however, the calculated curve agrees very well with the results of Murgatroyd and actually coincides over the inhibition region—within, say, the precision of the slide-rule which is the best we may hope for in view of the difficulties inherent to this semi-graphic method—with the curve tentatively drawn by Murgatroyd to connect his experimental points. The calculated curve diverges from these points upwards roughly where they cease to be on a single curve but we could not expect that any of our speculations would be very accurate at these rather low values of the Hartmann number and, in particular, the formula (41), as discussed above, presumably loses all significance as soon as the magnetic field ceases to be prevailing. (This, we believe, is roughly indicated by the greater sensitivity to the Reynolds number which appears for $10^4 M/R \leqslant 20$ (see figure 1 and also Nihoul 1966.) Approximately for the same value of M/R, the calculated points also fall into different curves (for different values of R) as Z is a function of M/R only—as pointed out before—provided $M^2/R \gg 1$ (dashed curves in figure 1).

Appendix

Substituting (27) into (26), we get

$$g = \sum_{n=0}^{\frac{1}{2}n_0} \sum_{K=0}^{S} \frac{4n+1}{n_0^{2K+1}} a_K P_{2n} [2n(2n+1)]^K$$

$$= \sum_0^S \frac{a_K}{n_0^{2K+1}} \sum_0^{\frac{1}{2}n_0} (4n+1)\,[2n(2n+1)]^K P_{2n} = \sum_0^S \frac{A_K a_K}{n_0^{2K+1}}.$$

The Legendre polynomials satisfy the relations (Tricomi 1955)

$$(4n+1)P_{2n} = P'_{2n+1} - P'_{2n-1}$$

and
$$(1-y^2)P''_{2n} - 2yP'_{2n} + 2n(2n+1)P_{2n} = 0.$$

Hence
$$A_0 = P'_{n_0+1} = C^{\frac{3}{2}}_{n_0} \quad \text{and} \quad A_m = [(y^2-1)A'_{m-1}]'.$$

Substituting for A_{m-1}, A_{m-2} and so on, the following recurrence formula is found

$$A_m = \sum_0^m (-1)^p (2p+1)!\, C^{\frac{3}{2}+p}_{n_0} \Sigma \prod_{p+1}^{m-p} (n_0+2).(n_0+3); (n_0+2p+2).(n_0+2p+3),$$

where $\Sigma \prod_{p+1}^{m-p} (n_0+2).(n_0+3); (n_0+2p+2).(n_0+2p+3)$ denotes the sum of all products of degree $m-p$ that one can form with $p+1$ factors of the type $(n_0+2r+2)(n_0+2r+3)$. There are $m!/(m-p)!\,p!$ of them.

The function g in the core

In the core setting $y = \cos\theta$, we have

$$C^{\lambda}_{n_0} = \frac{(\frac{1}{2}n_0)^{\lambda-1}}{\Gamma(\lambda)(\sin\theta)^{\lambda}} \cos[(n_0+\lambda)\theta - \lambda\tfrac{1}{2}\pi].$$

Hence

$$\frac{A_m}{n_0^{2m+1}} \sim \sum_0^m \frac{(-1)^p n_0^{-p-1} 2^p m!}{(m-p)!} \frac{(\frac{1}{2}n_0)^{\frac{1}{2}} \cos[(n_0+p+\frac{3}{2})\theta - (p+\frac{3}{2})\frac{1}{2}\pi]}{\Gamma(\frac{3}{2})(\sin\theta)^{p+\frac{3}{2}}}.$$

For large n_0 the dominant contribution comes from the term $p = 0$. Hence

$$\frac{A_m}{n_0^{2m+1}} \sim \frac{C^{\frac{3}{2}}_{n_0}}{n_0}, \quad g_c = \sum_0^S \frac{a_K A_K}{n_0^{2K+1}} = \frac{\alpha}{n_0} C^{\frac{3}{2}}_{n_0}.$$

The function g in the transition regions

In the transition regions, setting $\zeta/n_0 = (2n_0 - p + 2)(1-y)/(3+y)$ we have (Tricomi 1955)

$$C^{\lambda}_{n_0} \sim \frac{\Gamma(n_0+2\lambda)}{\Gamma(2\lambda)\, n_0!}\left[1 - \frac{\zeta}{n_0^2}\right]^{-n_0-2\lambda} e^{-\zeta/n_0}\, \Phi(-n_0 - \lambda + \tfrac{1}{2}, \lambda + \tfrac{1}{2}, \zeta/n_0),$$

where Φ is the limit of the hypergeometric function

$$\Phi(-n_0 - \lambda + \tfrac{1}{2}, \lambda + \tfrac{1}{2}, \zeta/n_0) = \lim_{\eta\to\infty} F\left(-n_0 - \lambda + \tfrac{1}{2}, \eta; \lambda + \tfrac{1}{2}; \frac{\zeta}{n_0 \eta}\right)$$

$$\sim (\lambda - \tfrac{1}{2})!\, \zeta^{\frac{1}{2}(-\lambda-1)} J_{\lambda-\frac{1}{2}}(2\zeta^{\frac{1}{2}}) \quad \text{for large } n_0,$$

where J is the Bessel function. Hence

$$A_m = \sum_0^m (-1)^p (2p+1)! \sum \prod_{p+1}^{m-p} (n_0+2).(n_0+3);(n_0+2p+2).(n_0+2p+3)$$
$$\times \frac{\Gamma(n_0+2p+3)}{\Gamma(2p+3)\, n_0!}(p+1)!\, \zeta^{-\frac{1}{2}(p+1)} J_{p+1}(2\zeta^{\frac{1}{2}}).$$

It may be shown that the Bessel functions satisfy

$$\frac{J_{p+1}(2\zeta^{\frac{1}{2}})}{\zeta^{\frac{1}{2}(p+1)}} = \sum_0^p \frac{p!}{q!\,(p-q)!}\frac{J_{2p+2-2q} + J_{2p-2q}}{(2p+1-q)\ldots(p+1-q)}$$
$$= 2\sum_1^{p+1} \frac{(p+1)!}{(p-r+1)!\,(p+r+1)!} J_{2r} + \frac{J_0}{(p+1)!}.$$

Hence

$$\frac{A_m}{n_0^{2m+1}} \sim n_0 \sum_0^m (-1)^p \frac{m!}{(m-p)!}\left\{\frac{J_0}{2(p+1)!} + \sum_1^{p+1}\frac{(p+1)!}{(p-r+1)!\,(p+r+1)!} J_{2r}\right\}$$
$$\sim n_0\left[\frac{J_0}{2(m+1)} + \frac{J_2}{(m+1)(m+2)} + \text{smaller terms}\right],$$

$$g_b = n_0[\beta J_0 + \gamma J_2 + \ldots],$$

with

$$\beta = \frac{1}{2}\sum_0^S \frac{a_K}{K+1}, \quad \gamma = \sum_0^S \frac{a_K}{(K+1)(K+2)}.$$

We restrict ourselves to the first two Bessel functions in this context, taking into account that $J_{2r}(2\zeta^{\frac{1}{2}}) \ll J_2(2\zeta^{\frac{1}{2}})$ for $r > 1$ and $\zeta \sim 1$.

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